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MANIFOLDS WITH PLANAR GEODESICS

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Theorem. *Let M be a connected submanifold of some Euclidean space; dimension $M \geq 2$. If every geodesic of M lies in a 2-plane, then M is either an open subset of an n-plane or is congruent to a dilatation of an open subset of S^n, RP^n, CP^n, QP^n or OP^2. Here S^n is the unit sphere and the others are particular submanifolds to be described.*

This paper is a continuation and in a sense a completion of the work of Sing-Long Hong [3]. Lemmas and propositions numbered 2 through 13 are essentially due to Hong. We have included them in some cases in order to clarify his work and in other cases to make our paper self-contained.

Denote by F either the real R, complex C, or quaternion Q, fields or the algebra of Cayley numbers O. On F the Euclidean inner product may be written $f_1 \cdot f_2 = \frac{1}{2}(f_1\bar{f}_2 + f_2\bar{f}_1)$, $f_1, f_2 \in F$. Let $M^n(F)$ be the $n \times n$ matrices over F. It is a Euclidean space with inner product $M_1 \cdot M_2 = \frac{1}{2}$ trace $(M_1\bar{M}_2^t + M_2\bar{M}_1^t)$ where M_i^t $(i = 1, 2)$ is the transpose of the matrix M_i. *The manifolds FP^n listed in the theorem may be defined as follows: $FP^n = \{M \in M^{n+1}(F) \mid M = \bar{M}^t$, $M = M^2$, and rank $M = 1\}$.* Note that when F is O we only define OP^2.

When F is R, C or Q it is well known that the manifolds given are embeddings of the abstractly defined projective spaces FP^n. In the case of the Cayley plane OP^2, one often takes this as the definition. It is also an embedded submanifold of Euclidean space.

Proposition 1. *The submanifolds of RP^n, CP^n, QP^n and OP^2 given above all have planar geodesics.*

Proof. Let F be R, C or Q. Any Hermitian symmetric matrix over F can be put in diagonal form by a change of basis. The diagonal form of a rank 1 matrix has a zero everywhere except for one element on the diagonal. Thus any Hermitian symmetric rank 1 matrix over F can be written $(f_i\bar{f}_j)$ for $f_i \in F$, $1 \leq i \leq n+1$. $\phi: F^{n+1} \to M^{n+1}(F)$, defined by $\phi(f_1, \cdots, f_{n+1}) = (f_i\bar{f}_j)$, maps F^{n+1} onto the Hermitian symmetric rank 1 matrices. For a matrix $M = (f_i\bar{f}_j)$ a simple computation shows that $M^2 = (\text{trace } M)M$. Hence $M = \phi(f_1, \cdots, f_{n+1})$ satisfies $M^2 = M$ if and only if trace $(f_i\bar{f}_j) = 1$, which is true if and only if $(f_1, \cdots, f_{n+1})$ lies on the unit sphere in F^{n+1}. Thus ϕ maps the unit sphere in F^{n+1} onto the previously defined FP^n. Also $\phi(f_1, \cdots, f_{n+1}) = \phi(f_1 w, \cdots, f_{n+1} w)$ for any unit vector w in F. Hence ϕ maybe defined on the abstract projective space

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over F, ϕ: $FP^n \to M^{n+1}(F)$. ϕ is an embedding of the abstract FP^n onto the embedded submanifolds previously defined. If $A : F^{n+1} \to F^{n+1}$ is a linear transformation such that $A\bar{A}^t = \bar{A}^t A = I$ we say A is orthogonal (for F). We may check that $\varphi(Av) = A\varphi(v)\bar{A}^t$ for $v \in F^{n+1}$. The mapping which sends $M \in M^{n+1}(F)$ to $AM\bar{A}^t$, where A is orthogonal, preserves the inner product in $M^{n+1}(F)$ and so is a Euclidean motion. Now the orthogonal transformations on F^{n+1} give projective transformations on FP^n. Hence the equation $\varphi(Av) = A\varphi(v)\bar{A}^t$ shows that any projective transformation of $\varphi(FP^n)$ arising from an orthogonal transformation of F^{n+1} can be accomplished by a Euclidean motion of $M^{n+1}(F)$. For this reason φ is said to be equivariant. The identity $\sum_{i,j} (f_i\bar{f}_j)\overline{(f_i\bar{f}_j)} = (\sum_i f_i\bar{f}_i)^2$ and the fact that $\sum_i f_i\bar{f}_i = 1$ show that $\varphi(FP^n)$ lies on the unit sphere about the origin in $M^{n+1}(F)$.

A projective line in the embedded manifold is a sphere of dimension 1, 2, or 4 according as F is R, C, or Q. It suffices using the equivariance to check this for just one projective line, say $\varphi(f_1, f_2, 0, \cdots, 0)$. Let $M = (m_{ij})$ be the coordinates in $M^{n+1}(F)$. Then $m_{11} = |f_1|^2$, $m_{12} = f_1\bar{f}_2$, $m_{21} = f_2\bar{f}_1$, $m_{22} = |f_2|^2$, the other $m_{ij} = 0$ and $|f_1|^2 + |f_2|^2 = 1$. So within the linear space $m_{ij} = 0$ for i, j not both 1 or 2, the projective line is the intersection of the sphere

$$|m_{12}|^2 + |m_{21}|^2 + |m_{11} - \tfrac{1}{2}|^2 + |m_{22} - \tfrac{1}{2}|^2 = \tfrac{1}{2}$$

with the linear spaces $m_{11} = \bar{m}_{11}$, $m_{22} = \bar{m}_{22}$, $m_{12} = \bar{m}_{21}$, $m_{11} - \frac{1}{2} = -(m_{22} - \frac{1}{2})$. These linear spaces pass through the center of the above sphere so that the projective line is a sphere of radius $1/\sqrt{2}$.

Since any pair of points lie on a projective line, all the projective lines, i.e., real spheres, through a given point cover all of $\varphi(FP^n)$.

Geodesics of $\varphi(FP^n)$ are the great circles of the projective lines (i.e., real spheres). To see this it suffices to show that a line from the center of any sphere to any point on the sphere meets $\varphi(FP^n)$ normally at that point. By equivariance it suffices to show this for one particular point and one particular projective line through that point.

Let the point be $P = \varphi(1, 0, \cdots, 0)$. Let $L_i = \varphi(f_1, 0, \cdots, 0, f_i, 0, \cdots, 0)$, $i = 2, \cdots, n+1$, be a set of projective lines through P. Then the tangent planes of L_i (as real spheres) span (and in fact give a direct sum decomposition of) the tangent space of $\varphi(FP^n)$ at P.

Let span L_i be the plane spanned by L_i, and let T be the tangent to the unit sphere about the origin (which contains $\varphi(FP^n)$) at P. It is not difficult to check that $T \cap \operatorname{span} L_i$ are completely orthogonal spaces meeting just at P. Thus the line from P to the center of L_2 is normal to $T \cap \operatorname{span} L_i$. (Consider the components along T and normal to T.) But $T \cap \operatorname{span} L_i$ contains the tangent plane to L_i at P. Hence the line from the center of L_2 to P meets each L_i orthogonally at P, and so it meets $\varphi(FP^n)$ orthogonally at P.

As for OP^2, the Cayley plane, consider first the 3×3 Hermitian matrices over O. They are of the form

$$M = \bar{M}^t = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = \begin{pmatrix} a_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & a_2 & x_1 \\ x_2 & \bar{x}_1 & a_3 \end{pmatrix},$$

where a_i are real and $x_i \in \boldsymbol{O}$. They form a Jordan algebra J with Jordan product $M_1 \cdot M_2 = \frac{1}{2}(M_1M_2 + M_2M_1)$. The group of automorphisms of J is a real form of an exceptional Lie group F_4. $\boldsymbol{O}P^2$ is the set of rank 1 matrices of J such that $M^2 = M$. Defining equations are $a_{ij} = x_k\bar{x}_k$, $a_k\bar{x}_k = x_ix_j$, $a_1 + a_2 + a_3 = 1$, for $(i, j, k) = (1, 2, 3)$, $(2, 3, 1)$ or $(3, 1, 2)$. F_4 acts transitively on pairs of polar points. Points M_1, M_2 are polar if trace $(M_1M_2 + M_2M_1) = 0$. For any point M_1 there is a projective line, the polar line, which is the locus of all points M_2 such that M_1M_2 are a polar pair. J has real dimension 27 and $\boldsymbol{O}P^2$ real dimension 16. For the above material concerning $\boldsymbol{O}P^2$ see Freudenthal [1].

Using the defining equations of $\boldsymbol{O}P^2$ we see that $\sum_{i,j} m_{ij}\bar{m}_{ij} = (a_1 + a_2 + a_3)^2 = 1$. Hence $\boldsymbol{O}P^2$ lies on the unit sphere in J about the origin.

For $\varphi \in F_4$ we have $\frac{1}{2}(\varphi(M_1)\varphi(M_2) + \varphi(M_2)\varphi(M_1)) = \frac{1}{2}(M_1M_2 + M_2M_1)$ because φ is a Jordan algebra automorphism. Hence it is surely true that trace $(\varphi(M_1)\varphi(M_2) + \varphi(M_2)\varphi(M_1))$ = trace $(M_1M_2 + M_2M_1)$. Hence F_4 preserves polarity, i.e., sends polar points into polar points. Now J, as a set of Hermitian symmetric matrices, is a linear subspace of $M^3(\boldsymbol{O})$. On J the Euclidean inner product may be written $M_1 \cdot M_2 = \frac{1}{2}$ trace $(M_1M_2 + M_2M_1)$ because $M = \bar{M}^t$ on J. Hence the elements of F_4 are Euclidean motions on J.

Because F_4 is transitive on polar pairs of points, it is also transitive on "pointed" projective lines. Namely, if L_1, L_2 are any pair of projective lines, and $P_1 \in L_1$, $P_2 \in L_2$ are points on those lines, then there is an element of F_4 sending P_1 to P_2 and L_1 to L_2. Let P_1' be the polar of L_1, and P_2' the polar of L_2. Then the required element of F_4 is the element sending the polar pair P_1P_1' to P_2P_2'.

Using the defining equations of $\boldsymbol{O}P^2$ we see that the polar line of the point $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the line $m_{11} = a_1$, $m_{12} = x_3$, $m_{21} = \bar{x}_3$, $m_{22} = a_2$, the other $m_{ij} = 0$, and $a_1 + a_2 = 1$, $a_1a_2 = x_3\bar{x}_3$. As before the projective line is the intersection of the sphere

$$|m_{12}|^2 + |m_{21}|^2 + |m_{11} - \tfrac{1}{2}|^2 + |m_{22} - \tfrac{1}{2}|^2 = \tfrac{1}{2}$$

with the linear spaces $m_{11} = \bar{m}_{11}$, $m_{22} = \bar{m}_{22}$, $m_{12} = \bar{m}_{21}$, $m_{11} - \frac{1}{2} = -(m_{22} - \frac{1}{2})$. Hence the projective line is a real 8-sphere of radius $1/\sqrt{2}$. Thus because F_4 is transitive on projective lines, every projective line is a real 8-sphere of radius $1/\sqrt{2}$.

The geodesics of $\boldsymbol{O}P^2$ are the great circles of its projective lines. As before it is enough to show that for any projective line L and any point P on L, the line from P to the center of L, as a real 8-sphere, is a normal line to OP^2 at P. Because F_4 is transitive on "pointed" projective lines, it is enough to show this when P is the point $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and L is the line $\begin{pmatrix} a_1 & x_3 & 0 \\ \bar{x}_3 & a_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $a_1 + a_2 = 1$, $a_1 a_2 = x_3 \bar{x}_3$. Let P' be the polar of L and L' the line joining P' and P, and T the tangent to the unit sphere with the origin as center at P. Then it is not difficult to show that $T \cap \operatorname{span} L$ and $T \cap \operatorname{span} L'$ are completely orthogonal spaces meeting just at P. Thus the line from P to the center of L must be orthogonal to the tangent planes of L and L' (as real 8-spheres) at P and hence orthogonal to the tangent plane of OP^2 at P. This completes the proof.

Let $p \in M$, and let γ be a curve on M with tangent vector t at p. Then the component of the second derivative of γ normal to M at p we call $\eta(t)$. (It is well known that this normal component depends only on t and not on the specific parametrized curve γ.) Thus $\eta: T_p \to N_p$ gives a map from the tangent space of M at p to the normal space of M at p, and this map is in fact a quadratic form. We will also use η to denote the associated bilinear form $\eta: T_p \times T_p \to N_p$, (so that $\eta(t, t) = \eta(t)$). We call η (in either sense) *the second fundamental form of M at P.*

Proposition 2. *If all the geodesics through a point of M are planar, then all those geodesics have the same curvature at that point. Here curvature means as a plane curve, not geodesic curvature.*

Proof. Let p be the point through which all geodesics are planar. We first show that $\eta(l_1)\cdot\eta(l_1, l_2) = 0$ for any orthonormal pair of tangent vectors l_1, l_2 at p. Let $\gamma(s)$ be a geodesic through p in the direction l_1, s the arc length from p, and let $l_2(s)$ be a parallel (in sense of Levi-Civita) tangent field to M along γ and normal to γ such that $l_2(0) = l_2$. Then $\eta(l_1) = d^2\gamma/ds^2(0)$ and $\eta(l_1, l_2) = dl_2/ds(0)$. Since γ is a geodesic, $d^2\gamma/ds^2$ is normal and therefore $d^2\gamma/ds^2 \cdot l_2 = 0$. If $d^2\gamma/ds^2(0) \neq 0$, then we may write $d^3\gamma/ds^3 = a\, d^2\gamma/ds^2 + b\, d\gamma/ds$. Thus $d^3\gamma/ds^3 \cdot l_2 = 0$. Now $0 = d/ds(d^2\gamma/ds^2 \cdot l_2) = d^3\gamma/ds^3 \cdot l_2 + d^2\gamma/ds^2 \cdot dl_2/ds$. Hence $d^2\gamma/ds^2 \cdot dl_2/ds = 0$ so that $\eta(l_1)\cdot\eta(l_1, l_2) = 0$.

As this is true for any orthonormal pair $l_1 l_2$, we must have

$$\eta(l_1 \cos\theta + l_2 \sin\theta)\cdot\eta(l_1 \cos\theta + l_2 \sin\theta,\ -l_1 \sin\theta + l_2 \cos\theta) = 0$$

for all θ. From this, using the bilinearity of η and double angle formulas we obtain

$$\tfrac{1}{2}(\eta(l_1, l_2)^2 - \tfrac{1}{4}(\eta(l_2) - \eta(l_1))^2) \sin 4\theta + \tfrac{1}{4}(\eta(l_2)^2 - \eta(l_1)^2) \sin 2\theta = 0 \ .$$

Hence $\eta(l_1, l_2)^2 - \frac{1}{4}(\eta(l_2) - \eta(l_1))^2 = 0$ and $\eta(l_1)^2 = \eta(l_2)^2$.

Now using the bilinearity and double angle formulas again

$$(\eta(l_1 \cos\theta + l_2 \sin\theta))^2 = (\tfrac{1}{4}(\eta(l_1) - \eta(l_2))^2 - \eta(l_1, l_2)^2)\cos 4\theta + \tfrac{1}{4}(\eta(l_1)^2 - \eta(l_2)^2)\cos 2\theta + (\tfrac{1}{2}(\eta(l_1) + \eta(l_2)))^2 + \tfrac{1}{2}(\tfrac{1}{4}(\eta(l_1) - \eta(l_2))^2 + \eta(l_1, l_2)^2) .$$

Hence η^2 is constant for all unit vectors in the plane of $l_1 l_2$.

Finally given any unit vectors l_1, l_2, not necessarily orthogonal, η^2 is constant on all unit vectors in their plane, so in particular $\eta^2(l_1) = \eta^2(l_2)$.

To finish we note that $|\eta(l_1)|$, l_1 a unit tangent vector, is the curvature of the geodesic through p in the direction l_1 at p.

Proposition 3. *Let $\gamma(t)$ be a curve of M, and $\gamma_t(s)$ a 1-parameter family of geodesics of M passing normally through γ, that is, $\gamma_t(0) = \gamma(t)$ and $d\gamma_t/ds(0) \cdot d\gamma/dt(t) = 0$. If the geodesics γ_t are planar, then they all have the same curvature as they cross γ, that is, if s is the arc length then $|d^2\gamma_t/ds^2(0)|$ is constant in t.*

Proof. Let $X(s, t) = \gamma_t(s)$ be considered as a surface in M. If we prove the curvature is constant in neighborhoods of points where $d\gamma_t/ds$, $d^2\gamma_t/ds^2$ are independent, that will suffice because the constant will be nonzero. Hence the intervals where $d\gamma_t/ds$, $d^2\gamma_t/ds^2$ are independent will be both open and closed and so all of γ. If there is no point on γ where $d\gamma_t/ds$, $d^2\gamma_t/ds^2$ are independent, then of course the result is true.

Now since $X(s, t)$ is a geodesic parametrized by the arc length for fixed t, we see that X_s is a unit tangent vector, i.e., $X_s \cdot X_s = 1$. By differentiating with respect to t we find that $X_s \cdot X_{st} = 0$. Next $(\partial/\partial s)(X_s \cdot X_t) = X_s \cdot X_{st} + X_{ss} \cdot X_t$. But since $X(s, t)$ is a geodesic for fixed t, X_{ss} is normal so $X_{ss} \cdot X_t = 0$. Thus $(\partial/\partial s)(X_s \cdot X_t) = 0$, and since $X_s \cdot X_t = 0$ for $s = 0$, it holds for all s, t.

Because the t held constant curves are planar, we may write $X_{sss} = \alpha X_s + \beta X_{ss}$ at a point where X_s, X_{ss} are independent. So using the above we have $X_{sss} \cdot X_t = 0$. Differentiating $X_{ss} \cdot X_t = 0$ with respect to s and using $X_{sss} \cdot X_t = 0$ we obtain $X_{ss} \cdot X_{st} = 0$. Again because $X_{sss} = \alpha X_s + \beta X_{ss}$ we have

$$X_{sss} \cdot X_{st} = \alpha X_s \cdot X_{st} + \beta X_{ss} \cdot X_{st} = 0 .$$

Differentiating $X_{ss} \cdot X_{st}$ with respect to s and using $X_{sss} \cdot X_t = 0$ we see that $X_{ss} \cdot X_{sst} = 0$. Hence $(\partial/\partial t)(X_{ss} \cdot X_{ss}) = 2X_{ss} \cdot X_{sst} = 0$. This implies that $(X_{ss}(0, t))^2$, which is the square of the curvature of γ_t at the point where it crosses γ, is constant.

Proposition 4. *If all the geodesics of M are planar, then either M^n is contained in an n-plane or else all the geodesics are circles of the same radius.*

Proof. Let $g(p)$ be the curvature of any geodesic passing through p at p. By Proposition 2, g is well defined. By Proposition 3, g is constant along curves and hence constant on M. Thus each geodesic has constant curvature and so is either a line or a circle. Furthermore all geodesics have the same curvature, so they are either all lines or all circles of the same radius.

We now suppose that M^n is not contained in an n-plane. We perform a dilatation of the Euclidean space to make all the geodesics circles of radius 1.

We see that for manifolds all of whose geodesics are circles of radius 1, $\eta^2(l_1) = 1$ for any unit tangent vector l_1. From this, using the fact that $\eta(\lambda l_1) = \lambda^2\eta(l_1)$ we have $\eta^2(t) = (t^2)^2$ for any tangent vector t.

Lemma 5. *$\eta(t)^2 = (t^2)^2$ for any tangent vector t has the following implications. For any orthonormal pair $l_1 l_2$*

$$\eta(l_1)\cdot\eta(l_1, l_2) = 0\ , \qquad \eta(l_1)\cdot\eta(l_2) + 2\eta(l_1, l_2)^2 = 1\ ,$$

for any orthonormal triple $l_1 l_2 l_3$

$$\eta(l_1)\cdot\eta(l_2, l_3) + 2\eta(l_1, l_2)\cdot\eta(l_1, l_3) = 0\ ,$$

and for any orthonormal quadruple $l_1 l_2 l_3 l_4$

$$\eta(l_1, l_2)\cdot\eta(l_3, l_4) + \eta(l_1, l_3)\cdot\eta(l_2, l_4) + \eta(l_1, l_4)\cdot\eta(l_2, l_3) = 0\ .$$

Of course the statements can only be made if the dimension is appropriate (i.e., dimension ≥ 4 for quadruple, etc.).

Proof. Let $t = x_1 l_1 + x_2 l_2 + x_3 l_3 + x_4 l_4$, ($x_4 = 0$ for dimension ≤ 3, etc.) Then $t^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, and $\eta(t) = \sum_{i,j=1}^4 x_i x_j \cdot \eta(l_i, l_j)$ by the bilinearity. Hence

$$\left(\sum_{i,j=1}^{4} x_i x_j \eta(l_i l_j)\right)^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^2\ .$$

Equating coefficients gives the result.

Lemma 6. *Let l_1, l_2 be orthonormal vectors with the property that $\eta(l_1)\cdot\eta(l_2, l_3) = 0$ for any unit vector l_3 normal to l_1 and l_2. Then $\eta(l_1)\cdot\eta(l_2) = \frac{1}{2}$ or 1.*

Proof. Since geodesics are circles of radius 1, the manifold may be written

$$X(r, l_1) = X(p) + (1 - \cos r)\eta(l_1) + l_1 \sin r\ ,$$

where r, l_1 are geodesic polar coordinates, and $l_1 \in TS_p^{n-1}$ is a unit tangent vector at p. Let $l_2, \cdots, l_n$ be orthonormal vectors, normal to l_1, defined in some neighborhood on TS_p^{n-1}.

$$\begin{aligned}\eta_{l_i}(l_1) &= \frac{d}{d\theta}\eta(l_1 \cos\theta + l_i \sin\theta)\,|_{\theta=0} \\ &= \frac{d}{d\theta}((\cos^2\theta)\eta(l_1) + 2\,(\cos\theta\sin\theta)\eta(l_1, l_i) + (\sin^2\theta)\eta(l_i))\,|_{\theta=0} \\ &= 2\eta(l_1, l_i) \qquad \text{for } i = 2, \cdots, n\ .\end{aligned}$$

$$\eta_{l_2l_2}(l_1) = \frac{d^2}{d\theta^2}\eta(l_1 \cos\theta + l_2 \sin\theta)\,|_{\theta=0}$$
$$= \frac{d^2}{d\theta^2}((\cos^2\theta)\eta(l_1) + 2(\cos\theta\sin\theta)\eta(l_1, l_2) + (\sin^2\theta)\eta(l_2))\,|_{\theta=0}$$
$$= 2(\eta(l_2) - \eta(l_1)) .$$

Hence

$$X_{l_i}(r, l_1) = (1 - \cos r)\eta_{l_i}(l_1) + l_{1l_i}\sin r$$
$$= 2(1 - \cos r)\eta(l_1, l_i) + l_i \sin r ,$$
$$X_{l_2l_2}(r, l_1) = (1 - \cos r)\eta_{l_2l_2}(l_1) + l_{1l_2l_2}\sin r$$
$$= 2(1 - \cos r)(\eta(l_2) - \eta(l_1)) - l_1 \sin r ,$$
$$X_r(r, l_1) = (\sin r)\eta(l_1) + l_1 \cos r .$$

$X_{l_i}\cdot X_r = 0$ for $i = 2, \cdots, n$ because $\eta(l_1)\cdot\eta(l_1, l_i) = 0$ by Lemma 5. So $X_{l_2l_2}\cdot X_{l_i} = 4(1 - \cos r)^2\eta(l_1, l_i)\cdot(\eta(l_2) - \eta(l_1))$, $i = 2, \cdots, n$. By Lemma 5, $\eta(l_1)\cdot\eta(l_1, l_i) = 0$ for $i = 2, \cdots, n$ and $\eta(l_1, l_2)\cdot\eta(l_2) = 0$. Thus, if $\eta(l_2)\cdot\eta(l_1, l_i) = 0$ for $i = 3, \cdots, n$ we have $X_{l_2l_2}\cdot X_{l_i} = 0$. Since the conclusion of the lemma is symmetric in l_1 and l_2, we may interchange the roles of l_1 and l_2 throughout the proof. We then require that $\eta(l_1)\cdot\eta(l_2, l_i) = 0$ for $i = 3, \cdots, n$, which is the hypothesis. Hence $X_{l_2l_2}\cdot X_{l_i} = 0$.

$$X_{l_2l_2}\cdot X_r = 2(1 - \cos r)(\sin r)\eta(l_1)\cdot(\eta(l_2) - \eta(l_1)) - \sin r \cos r .$$

Also $X_r\cdot X_r = 1$ because r is the arc length. Hence $X^N_{l_2l_2} = X_{l_2l_2} - (X_{l_2l_2}\cdot X_r)X_r$. ($N$ means normal component.) Now $\eta(t)^2 = (t^2)^2$ for any tangent vector t. When $t = X_{l_2}(r, l_1)$ we see that $\eta(t) = X^N_{l_2l_2}$. Thus $(X^N_{l_2l_2})^2 - (X^2_{l_2})^2 = 0$. But $X^N_{l_2l_2} = X_{l_2l_2} - (X_{l_2l_2}\cdot X_r)X_r$, which implies $(X^N_{l_2l_2})^2 = X^2_{l_2l_2} - (X_{l_2l_2}\cdot X_r)^2$. From above computations

$$X^2_{l_2l_2} = 4(1 - \cos r)^2(\eta(l_2) - \eta(l_1))^2 + \sin^2 r ,$$
$$X^2_{l_2} = 4(1 - \cos r)^2\eta(l_1, l_2)^2 + \sin^2 r .$$

From Lemma 5, $\eta(l_1)\cdot\eta(l_2) + 2\eta(l_1, l_2)^2 = 1$ so

$$X^2_{l_2} = 2(1 - \cos r)^2(1 - \eta(l_1)\cdot\eta(l_2)) + \sin^2 r .$$

Thus

$$0 = X^2_{l_2l_2} - (X_{l_2l_2}\cdot X_r)^2 - (X^2_{l_2})^2$$
$$= 4(1 - \cos r)^2(\eta(l_2) - \eta(l_1))^2 + \sin^2 r$$
$$- (2(1 - \cos r)\sin r(\eta(l_2)\cdot\eta(l_1) - 1) - \sin r\cos r)^2$$
$$- (2(1 - \cos r)^2(1 - \eta(l_1)\cdot\eta(l_2)) + \sin^2 r)^2 .$$

This after some simplification gives

$$0 = 4(1 - \cos r)^3(1 - \eta(l_1)\cdot\eta(l_2))(2\eta(l_1)\cdot\eta(l_2) - 1) ,$$

which must hold for all r. This concludes the proof.

For any unit tangent vector l_1 let $\alpha(l_1) = \{t \in T_p \mid \eta(t/|t|) = \eta(l_1) \text{ or } t = 0\}$.

Proposition 7. *$\alpha(l_1)$ is a linear subspace of T_p.*

Proof. Suppose l_2 is a unit vector such that $l_1 \wedge l_2 \neq 0$. Let l_3 be a unit vector in the plane of $l_1 l_2$ and normal to l_1. We may write $l_2 = al_1 + bl_3$, $a^2 + b^2 = 1$, $b \neq 0$. Then

$$\eta(l_2) = a^2\eta(l_1) + 2ab\eta(l_1, l_3) + b^2\eta(l_3) .$$

By Lemma 5, $\eta(l_1)\cdot\eta(l_1, l_3) = 0$ so $\eta(l_1)\cdot\eta(l_2) = a^2 + b^2\eta(l_1)\cdot\eta(l_3)$. Because $a^2 + b^2 = 1$, $b \neq 0$ we see that $\eta(l_1)\cdot\eta(l_2) = 1$ if and only if $\eta(l_1)\cdot\eta(l_3) = 1$. Hence $l_2 \in \alpha(l_1)$ if and only if $l_3 \in \alpha(l_1)$. Thus, if any tangent vector $t \in \alpha(l_1)$ then $\operatorname{span}(t, l_1) \subset \alpha(l_1)$. Hence it suffices to show that the vectors in $\alpha(l_1)$, which are orthogonal to l_1, are a linear subspace of T_p.

So suppose $l_2, l_3 \in \alpha(l_1)$ are unit vectors and $l_2\cdot l_1 = l_3\cdot l_1 = 0$. Since $l_2, l_3 \in \alpha(l_1)$, we have $\eta(l_2)\cdot\eta(l_1) = \eta(l_3)\cdot\eta(l_1) = 1$ and therefore $\eta(l_2, l_1) = \eta(l_3, l_1) = 0$ by Lemma 5. Thus $\eta(al_2 + bl_3, l_1) = a\eta(l_2, l_1) + b\eta(l_3, l_1) = 0$. Let $l_4 = (al_2 + bl_3)/|al_2 + bl_3|$. Then $\eta(l_4, l_1) = 0$ and $l_1 l_4$ are an orthonormal pair. Thus by Lemma 5, $\eta(l_4)\cdot\eta(l_1) = 1$ so that $l_4 \in \alpha(l_1)$. Hence $al_2 + bl_3 \in \alpha(l_1)$ for any a, b, which concludes the proof.

Remark. If X is a point of M and l_1 a unit tangent vector, then the geodesic through X in the direction l_1 is centered at $X + \eta(l_1)$. Thus all geodesics through X tangent to $\alpha(l_1)$ have the same center. Thus all geodesics through a point, which have the same center, fill out a sphere.

Let $S(l_1)$ be the unit vectors in $\alpha(l_1)^\perp$, the orthogonal complement of $\alpha(l_1)$. Let $f_{l_1}: S(l_1) \to \boldsymbol{R}$ be defined by $f_{l_1}(l) = \eta(l_1)\cdot\eta(l)$.

Lemma 8. *Let l_2 be a critical point of f_{l_1}. Then*

$$\eta(l_1)\cdot\eta(l_2, l_3) = 0$$

for all unit vectors l_3 orthogonal to l_1 and l_2.

Proof. Suppose $l_3 \in \alpha(l_1)$, l_3 a unit vector. Then $\eta(l_1) = \eta(l_3)$, which implies $\eta(l_1)\cdot\eta(l_3) = 1$. Using Lemma 5 we have $\eta(l_1)\cdot\eta(l_3) + 2\eta(l_1, l_3)^2 = 1$ so that $\eta(l_1, l_3) = 0$. Again by Lemma 5, $\eta(l_1)\cdot\eta(l_2\cdot l_3) + 2\eta(l_1, l_2)\cdot\eta(l_1, l_3) = 0$. Hence $\eta(l_1)\cdot\eta(l_2, l_3) = 0$.

Suppose $l_3 \in \alpha(l_1)^\perp$. Then the derivative of $f_{l_1}(l_2\cos\theta + l_3\sin\theta)$ with respect to θ at $\theta = 0$ is 0 because l_2 is a critical point of f_{l_1}.

$$\begin{aligned} f_{l_1}(l_2\cos\theta + l_3\sin\theta) &= \eta(l_1)\cdot\eta(l_2\cos\theta + l_3\sin\theta) \\ &= \eta(l_1)\cdot((\cos^2\theta)\eta(l_2) + 2(\cos\theta\sin\theta)\eta(l_2, l_3) + (\sin^2\theta)\eta(l_3)) . \end{aligned}$$

So $0 = df_{l_1}/d\theta|_{\theta=0} = 2\eta(l_1)\cdot\eta(l_2, l_3)$.

Now in general any l_3 may be written $l_3 = l_4 \cos\theta + l_5 \sin\theta$ for $l_4 \in \alpha(l_1)$, $l_5 \in \alpha(l_1)^\perp$. Since $l_1\cdot l_3 = 0$ and $l_1\cdot l_5 = 0$, we must have $l_1\cdot l_4 = 0$. Since $l_2\cdot l_3 = l_2\cdot l_4 = 0$, we must have $l_2\cdot l_5 = 0$. Thus by the previous cases $\eta(l_1)\cdot\eta(l_2, l_i) = 0$, $i = 4, 5$. Hence

$$\eta(l_1)\cdot\eta(l_2, l_3) = (\cos\theta)\eta(l_1)\cdot\eta(l_2, l_4) + (\sin\theta)\eta(l_1)\cdot\eta(l_2, l_5) = 0 .$$

Lemma 9. *Let l_1, l_2 be orthonormal tangent vectors. Then $l_2 \in \alpha(l_1)^\perp$ if and only if $\eta(l_1)\cdot\eta(l_2) = \frac{1}{2}$.*

Proof. Suppose $l_2 \in \alpha(l_1)^\perp$. If l_2 is a critical point of f_{l_1}, then Lemma 6 and Lemma 8 show that $\eta(l_1)\cdot\eta(l_2) = \frac{1}{2}$ or 1. But $\eta(l_1)\cdot\eta(l_2) = 1$ implies $\eta(l_1) = \eta(l_2)$ and so $l_2 \in \alpha(l_1)$. So the assumption $l_2 \in \alpha(l_1)^\perp$ shows that $\eta(l_1)\cdot\eta(l_2) = \frac{1}{2}$. But the critical points of f_{l_1} include both its maximum and minimum points. Hence $\eta(l_1)\cdot\eta(l) = f_{l_1}(l) = \frac{1}{2}$ for all l in the domain of f_{l_1} which is all unit vectors in $\alpha(l_1)^\perp$.

Now suppose $\eta(l_1)\cdot\eta(l_2) = \frac{1}{2}$. Write $l_2 = al_3 + bl_4$, where $l_3 \in \alpha(l_1)$, $l_4 \in \alpha(l_1)^\perp$ and $a^2 + b^2 = 1$. Then $\eta(l_1) = \eta(l_3)$ and by the first part $\eta(l_1)\cdot\eta(l_4) = \frac{1}{2}$. Hence

$$\begin{aligned}\tfrac{1}{2} &= \eta(l_1)\cdot\eta(l_2) = \eta(l_3)\cdot\eta(l_2) = \eta(l_3)\cdot\eta(al_3 + bl_4)\\ &= \eta(l_3)\cdot(a^2\eta(l_3) + 2ab\eta(l_3, l_4) + b^2\eta(l_4)) = a^2 + \tfrac{1}{2}b^2 .\end{aligned}$$

Here $\eta(l_3)\cdot\eta(l_3, l_4) = 0$ by Lemma 5. So $\frac{1}{2} = a^2 + \frac{1}{2}b^2$ and $a^2 + b^2 = 1$, which give $a = 0$. Thus $l_2 \in \alpha(l_1)^\perp$.

We call a linear subspace L of T_p *closed with respect to* α if $l \in L$ implies $\alpha(l) \subset L$ for any unit vector l.

Lemma 10. *If L is closed with respect to α, then $L^\perp$, the orthogonal complement, is also closed with respect to α.*

Proof. Take $l_1 \in L^\perp$ and $l_2 \in \alpha(l_1)$. Then we may write $l_2 = al_3 + bl_4$, $l_3 \in L$, $l_4 \in L^\perp$, $a^2 + b^2 = 1$. $\eta(l_1) = \eta(l_2) = a^2\eta(l_3) + 2ab\eta(l_3, l_4) + b^2\eta(l_4)$. Since L is closed with respect to α, we have $\alpha(l_3) \subset L$, so that $l_1, l_4 \in \alpha(l_3)^\perp$. By Lemma 9, $\eta(l_1)\cdot\eta(l_3) = \eta(l_4)\cdot\eta(l_3) = \frac{1}{2}$. Thus $\frac{1}{2} = \eta(l_1)\cdot\eta(l_3) = a^2 + 2ab\eta(l_3)\cdot\eta(l_3, l_4) + \frac{1}{2}b^2$. By Lemma 5, $\eta(l_3)\cdot\eta(l_3, l_4) = 0$. So $\frac{1}{2} = a^2 + \frac{1}{2}b^2$, which together with $a^2 + b^2 = 1$ gives $a = 0$. Hence $l_2 \in L^\perp$.

Lemma 11. *Assume all the orthonormal vectors below satisfy $l_i \in \alpha(l_j)$ or $l_i \in \alpha(l_j)^\perp$ for any i, j, $i \neq j$. Then: for any unit vector*

$$\eta(l_1)^2 = 1 ;$$

for any orthonormal pair

$$\eta(l_1)\cdot\eta(l_2) = \begin{cases} 1 & \text{if } l_1 \in \alpha(l_2) , \\ \frac{1}{2} & \text{if } l_1 \in \alpha(l_2)^\perp , \end{cases}$$

$$\eta(l_1, l_2)^2 = \begin{cases} 0 & \textit{if } l_1 \in \alpha(l_2) \ , \\ \frac{1}{4} & \textit{if } l_1 \in \alpha(l_2)^\perp \ , \end{cases}$$

$$\eta(l_1) \cdot \eta(l_1, l_2) = 0 \ ;$$

for any orthonormal triple

$$\eta(l_1) \cdot \eta(l_2, l_3) = 0 \ , \qquad \eta(l_1, l_2) \cdot \eta(l_1, l_3) = 0 \ ;$$

for any orthonormal quadruple

$$\eta(l_1, l_2) \cdot \eta(l_3, l_4) = 0 \ ,$$

if $l_1 \in \alpha(l_2)$ *or* $l_3 \in (l_4)$, *or if* $l_i \in \alpha(l_j)^\perp$ *for all* i, j, $i \neq j$.

Notice that we have not covered all cases for an orthonormal quadruple of vectors.

Proof. $\eta(l_1)^2 = 1$ if l_1 is a unit vector because geodesics are circles of radius 1.

Let l_1, l_2 be orthonormal vectors satisfying the conditions of the lemma. If $l_1 \in \alpha(l_2)$, then $\eta(l_1) = \eta(l_2)$ so $\eta(l_1) \cdot \eta(l_2) = \eta(l_1)^2 = 1$. If $l_1 \in \alpha(l_2)^\perp$, then by Lemma 9, $\eta(l_1) \cdot \eta(l_2) = \frac{1}{2}$. Since $\eta(l_1) \cdot \eta(l_2) + 2\eta(l_1, l_2)^2 = 1$ by Lemma 5, $\eta(l_1, l_2)^2 = 0$ or $\frac{1}{4}$ according as $l_1 \in \alpha(l_2)$ or $l_1 \in \alpha(l_2)^\perp$. Also $\eta(l_1) \cdot \eta(l_1, l_2) = 0$ by Lemma 5.

Let $l_1 l_2 l_3$ be an orthonormal triple satisfying the conditions of the lemma. Assume $l_2 \in \alpha(l_1)^\perp$. From Lemma 9 we see that $\eta(l_1) \cdot \eta(l) = \frac{1}{2}$ for all unit vectors $l \in \alpha(l_1)^\perp$. Hence the function f_{l_1} of Lemma 8 is constant so that every point of its domain is a critical point. But since $l_2 \in \alpha(l_1)^\perp$, l_2 is in the domain of f_{l_1} and hence a critical point of f_{l_1}. Thus by Lemma 8, $\eta(l_1) \cdot \eta(l_2, l_3) = 0$. Next assume $l_2 \in \alpha(l_1)$. Use Lemma 5 to write

$$\eta(l_1) \cdot \eta(l_2, l_3) + 2\eta(l_1, l_2) \cdot \eta(l_1, l_3) = 0 \ .$$

From above if $l_2 \in \alpha(l_1)$ then $\eta(l_1, l_2) = 0$ so $\eta(l_1) \cdot \eta(l_2, l_3) = 0$. Now $\eta(l_1, l_2) \cdot \eta(l_1, l_3) = 0$ for any triple satisfying the conditions of the lemma by Lemma 5 and the fact that $\eta(l_1) \cdot \eta(l_2, l_3) = 0$.

Next let $l_1 l_2 l_3 l_4$ be an orthonormal quadruple such that $l_i \in \alpha(l_j)^\perp$ for $1 \leq i, j \leq 4$. In particular $l_1 l_2$ are in $\alpha(l_3)^\perp$ and $\alpha(l_4)^\perp$. Hence $(l_1 + l_2)/\sqrt{2}$ is in $\alpha(l_3)^\perp$ and $\alpha(l_4)^\perp$. Using Lemma 9 we see that $l_i \in \alpha(l_j)^\perp$ if and only if $l_j \in \alpha(l_i)^\perp$. Thus $(l_1 + l_2)/\sqrt{2}, l_3, l_4$ are an orthonormal triple satisfying the conditions of this lemma. Hence $\eta((l_1 + l_2)/\sqrt{2}) \cdot \eta(l_3, l_4) = 0$. Also since $l_1 l_3 l_4$ and $l_2 l_3 l_4$ are triples satisfying the conditions of this lemma, we have $\eta(l_1) \cdot \eta(l_3, l_4) = 0$ and $\eta(l_2) \cdot \eta(l_3 \cdot l_4) = 0$. So

$$\begin{aligned} 0 &= \eta((l_1 + l_2)/\sqrt{2}) \cdot \eta(l_3, l_4) \\ &= (\tfrac{1}{2}\eta(l_1) + \eta(l_1, l_2) + \tfrac{1}{2}\eta(l_2)) \cdot \eta(l_3, l_4) = \eta(l_1, l_2) \cdot \eta(l_3, l_4) \ . \end{aligned}$$

If $l_1 \in \alpha(l_2)$ then $\eta(l_1, l_2) = 0$, and if $l_3 \in \alpha(l_4)$ then $\eta(l_3, l_4) = 0$. Hence in these cases also $\eta(l_1, l_2) \cdot \eta(l_3, l_4) = 0$. This finishes the proof of Lemma 11.

Lemma 12. *If L_1 and L_2 are completely orthogonal subspaces of T_p both closed with respect to α, then their linear span is also closed with respect to α.*

Proof. Let $l \in \operatorname{span}(L_1, L_2)$ be a unit vector, and let l' be a unit vector in $\alpha(l)$, $l' \in \alpha(l)$. Then we may write

$$l' = al_1 + bl_2 + cl_3 ,$$

where $l_1 \in L_1$, $l_2 \in L_2$ and $l_3 \in \operatorname{span}(L_1, L_2)^\perp$ are unit vectors and $a^2 + b^2 + c^2 = 1$.

$$\begin{aligned}\eta(l') &= \eta(al_1 + bl_2 + cl_3) \\ &= a^2\eta(l_1) + b^2\eta(l_2) + c^2\eta(l_3) + 2ab\eta(l_1, l_2) \\ &\quad + 2ac\eta(l_1, l_3) + 2bc\eta(l_2, l_3) .\end{aligned}$$

Since L_1 and L_2 are closed with respect to α, $l_3 \in \alpha(l_1)^\perp$ and $l_3 \in \alpha(l_2)^\perp$. Thus $\eta(l_3) \cdot \eta(l_1) = \frac{1}{2}$ and $\eta(l_3) \cdot \eta(l_2) = \frac{1}{2}$. So

$$\eta(l') \cdot \eta(l_3) = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 + c^2 = \tfrac{1}{2} + \tfrac{1}{2}c^2 .$$

On the other hand $l \in \operatorname{span}(L_1, L_2)$ can be written $l = rl_4 + sl_5$ where $l_4 \in L_1$, $l_5 \in L_2$ are unit vectors and $r^2 + s^2 = 1$. Thus

$$\eta(l) = \eta(rl_4 + sl_5) = r^2\eta(l_4) + 2rs\eta(l_4, l_5) + s^2\eta(l_5) .$$

Again because L_1 and L_2 are closed with respect to α, we must have $l_3 \in \alpha(l_4)^\perp$ and $l_3 \in \alpha(l_5)^\perp$. Hence

$$\eta(l) \cdot \eta(l_3) = \tfrac{1}{2}r^2 + \tfrac{1}{2}s^2 = \tfrac{1}{2} .$$

But $\eta(l) = \eta(l')$ so $\frac{1}{2} = \frac{1}{2} + \frac{1}{2}c^2$ giving $c = 0$ and $l' = al_1 + bl_2$. Thus $l' \in \operatorname{span}(L_1, L_2)$ which concludes the lemma.

Proposition 13. *For any unit tangent vector l_1 at any point p, the dimension of $\alpha(l_1)$ is the same. We call it a.*

Proof. Let $a(l_1)$ be the dimension of $\alpha(l_1)$. We will show that

$$\eta(l_1) \cdot H = \tfrac{1}{2} + \tfrac{1}{2}a(l_1)/n ,$$

where H is the mean curvature vector. The result follows from this because $\eta(l_1) \cdot H$ is continuous on the unit tangent bundle and $a(l_1)$ is integer-valued.

Choose orthonormal tangent vectors $l_1, \cdots, l_n$ so that $l_1, \cdots, l_a$ span $\alpha(l_1)$. Then $H = (1/n) \sum_{i=1}^{n} \eta(l_i)$ so that

$$\eta(l_1) \cdot H = \frac{1}{n} \sum_{i=1}^{a} \eta(l_1) \cdot \eta(l_i) + \frac{1}{n} \sum_{i=a+1}^{n} \eta(l_1) \cdot \eta(l_i) .$$

Now $\eta(l_1) = \eta(l_i)$ for $i = 1, \cdots, a$ so $\eta(l_1)\cdot\eta(l_i) = 1$. For $i = a + 1, \cdots, n$, $l_i \in \alpha(l_1)^\perp$ we have $\eta(l_1)\cdot\eta(l_i) = \frac{1}{2}$ by Lemma 9. Hence

$$\eta(l_1)\cdot H = \frac{a}{n} + \frac{n-a}{2n} ,$$

which concludes the proof.

The quadratic form $\eta: T_p \to N_p$ sends a linear space of dimension a, say $\alpha(l)$, into a line, the line through $\eta(l)$. Hence the rank of the Jacobian of η must fall by $a - 1$ at every point of T_p.

Now if L is a linear space of T_p closed with respect to α, then the restriction $\eta: L \to N_p$ of η to L also sends linear spaces of dimension a into lines. *Hence the Jacobian of the restriction of η to a linear space closed with respect to α falls by $a - 1$ in rank.*

According to Lemma 10 if $l_2 \in \alpha(l_1)^\perp$ is a unit vector then $\alpha(l_2) \subset \alpha(l_1)^\perp$. We choose vectors $l_i \in \bigcap_{j=1}^{i-1} \alpha(l_j)^\perp$ by induction. This decomposes T_p into a direct sum

$$T_p = \alpha(l_1) \oplus \cdots \oplus \alpha(l_k) ,$$

where of course $\alpha(l_i) \subset \alpha(l_j)^\perp$, $i \neq j$.

Since the dimension of $\alpha(l_i)$ is a, we see that $ak = n$ so that *a divides n.*

Let us choose an orthonormal basis $l_1 \cdots l_n$ of T_p in agreement with the direct sum decomposition of T_p given above, namely, each basis vector is in one of the summands. Such a basis has the property that either $l_i \in \alpha(l_j)$ or $l_i \in \alpha(l_j)^\perp$ for any i, j, $i \neq j$. Any basis with this property we call *a basis which respects α.*

Lemma 14. *Suppose $a = 2$. Let L_1, L_2 be completely orthogonal α-closed subspaces of dimension 2. Let l_1l_2 be a basis of L_1 and l_3l_4 of L_2, both orthonormal. Then in this basis or the one obtained by reflection in L_2 (sending $l_3 \to -l_3$) we have*

$$\eta(l_1, l_3) = \eta(l_2, l_4) , \qquad \eta(l_1, l_4) = -\eta(l_2, l_3) .$$

Furthermore, if L_1, L_2, L_3 are completely orthogonal α-closed subspaces of dimension 2, and l_1l_2, l_3l_4, l_5l_6 are respective orthonomal bases such that the above relations hold on $L_1 \oplus L_2$ and $L_1 \oplus L_3$, then they also hold on $L_2 \oplus L_3$.

Proof. By Lemma 12 and the comment after Proposition 13 the restriction of the Jacobian of η to $L_1 \oplus L_2$ falls in rank by 1. The restriction is

$$\eta(x_1l_1 + \cdots + x_4l_4) = \sum_{i,j=1}^{4} x_ix_j\eta(l_i, l_j) ,$$

with derivatives

$$\eta_{x_i} = 2 \sum_{j=1}^{4} x_j \eta(l_i, l_j) ,$$

evaluated at $l_1 + l_3$, which are

$$\eta_{x_1} = 2(\eta(l_1) + \eta(l_1, l_3)) , \qquad \eta_{x_2} = 2\eta(l_2, l_3) ,$$
$$\eta_{x_3} = 2(\eta(l_3, l_1) + \eta(l_3)) , \qquad \eta_{x_4} = 2\eta(l_4, l_1) .$$

Because the Jacobian falls in rank by 1, these four vectors must be dependent. But $\eta(l_1), \eta(l_3)$ are orthogonal to $\eta(l_i, l_j)$, $i \neq j$, and independent. Hence we must have $\eta(l_2, l_3)$ and $\eta(l_4, l_1)$ linearly dependent. Since they are the same length, we must have $\eta(l_2, l_3) = \pm\eta(l_1, l_4)$. We now reverse the sign of l_3 if necessary to achieve $\eta(l_2, l_3) = -\eta(l_1, l_4)$. Use Lemma 5 to write

$$\eta(l_1, l_2)\cdot\eta(l_3, l_4) + \eta(l_1, l_3)\cdot\eta(l_2, l_4) + \eta(l_1, l_4)\cdot\eta(l_2, l_3) = 0 .$$

Because $\eta(l_1, l_2) = 0$ we have

$$\eta(l_1, l_3)\cdot\eta(l_2, l_4) = -\eta(l_1, l_4)\cdot\eta(l_2, l_3) .$$

But $\eta(l_i, l_j)$, $i = 1$ or 2, $j = 3$ or 4, are all the same length and $\eta(l_2, l_3) = -\eta(l_1, l_4)$. Hence $\eta(l_1, l_3) = \eta(l_2, l_4)$ and the first part of the lemma is completed.

Now this same argument applied to $L_1 \oplus L_3$ shows that (perhaps after sending l_5 to $-l_5$)

$$\eta(l_1, l_5) = \eta(l_2, l_6) , \qquad \eta(l_1, l_6) = -\eta(l_2, l_5) .$$

When we apply this argument to $L_2 \oplus L_3$ we find that

$$\eta(l_3, l_5) = \lambda\eta(l_4, l_6) , \qquad \eta(l_3, l_6) = -\lambda\eta(l_4, l_5) ,$$

where $\lambda = \pm 1$. We must show that $\lambda = +1$.

On $L_1 \oplus L_2 \oplus L_3$ the Jacobian of η falls in rank by 1. We evaluate the derivatives η_{x_i}, $i = 1, \cdots, 6$, at the point $l_1 + l_3 + l_5$. $\eta_{x_1}, \eta_{x_3}, \eta_{x_5}$ have respectively the term $\eta(l_1), \eta(l_3), \eta(l_5)$. Because these vectors are independent (they are of length 1 and the inner product of any two is $\frac{1}{2}$) and orthogonal to $\eta(l_i, l_j)$, $i \neq j$, we see, much as before, that $\eta_{x_2}, \eta_{x_4}, \eta_{x_6}$ given by

$$\eta_{x_2} = 2(\eta(l_2, l_3) + \eta(l_2, l_5)) ,$$
$$\eta_{x_4} = 2(\eta(l_4, l_1) + \eta(l_4, l_5)) ,$$
$$\eta_{x_6} = 2(\eta(l_6, l_1) + \eta(l_6, l_3)) ,$$

must be dependent. We use the above relations and those on $L_1 \oplus L_2$ to obtain

$$\eta_{x_2} = -2(\eta(l_1, l_4) + \eta(l_1, l_6)) ,$$
$$\eta_{x_4} = 2(\eta(l_1, l_4) - \lambda\eta(l_3, l_6)) ,$$

$$\eta_{x_6} = 2(\eta(l_1, l_6) + \eta(l_3, l_6)) \ .$$

Hence

$$0 = \eta_{x_2} \wedge \eta_{x_4} \wedge \eta_{x_6} = -8(1-\lambda)\eta(l_1, l_4) \wedge \eta(l_3, l_6) \wedge \eta(l_1, l_6) \ .$$

Using Lemma 11 and the fact that $\eta(l_3, l_6) = -\lambda\eta(l_4, l_5)$, $\lambda = \pm 1$ we see that $\eta(l_1, l_4)$, $\eta(l_3, l_6)$ and $\eta(l_1, l_6)$ are orthogonal. Because they are nonzero, they are independent and so $\lambda = +1$.

Remark. Quaternion multiplication on a basis $l_1l_2l_3l_4$ may be defined by $-l_jl_i = l_il_j = l_k$ for i, j, k any cyclic permutation of 2, 3, 4 and $l_il_1 = l_1l_i = l_i$ for all i and $l_i^2 = -l_1$ for $i = 2, 3, 4$. The conjugation is defined by $\bar{l}_1 = l_1$, $\bar{l}_i = -l_i$, $i = 2, 3, 4$.

Lemma 15. *Suppose $a = 4$. Let L_1, L_2 be two completely orthogonal subspaces of dimension 4, both closed with respect to α. Let $l_1l_2l_3l_4$ be an orthonormal basis of L_1. Then for either this basis or its reflection (sending $l_1 \to -l_1$) there is an orthonormal basis $l_5l_6l_7l_8$ of L_2 such that $\eta(l_i, l_{j+4}) = \pm\eta(l_k, l_{m+4})$ if and only if $l_i\bar{l}_j = \pm l_k\bar{l}_m$ in the quaternion multiplication. Here both signs are taken as positive or both negative and the indices range from 1 to 4.*

Proof. Let $l_1l_2l_3l_4$ be the given basis of L_1, and $l_5l_6l_7l_8$ any orthonormal basis of L_2. We may restrict η to $L_1 \oplus L_2$ and the Jacobian must still fall in rank by 3. The restriction is

$$\eta(x_1l_1 + \cdots + x_8l_8) = \sum_{i,j=1}^{8} x_ix_j\eta(l_i, l_j) \ .$$

We now compute the Jacobian of η at $l_k + l_5$, $1 \leq k \leq 4$. Since

$$\eta_{x_i} = 2\sum_{j=1}^{8} x_j\eta(l_i, l_j) \ ,$$

we have, at $l_k + l_5$,

$$\eta_{x_k} = 2\eta(l_k) + 2\eta(l_k, l_5)\,; \quad \eta_{x_i} = 2\eta(l_i, l_5) \ , \qquad 1 \leq i \leq 4,\ i \neq k\,;$$
$$\eta_{x_5} = 2\eta(l_5) + 2\eta(l_k, l_5)\,; \quad \eta_{x_i} = 2\eta(l_k, l_i) \ , \qquad i = 6, 7, 8 \ .$$

Now $\eta(l_k) = \eta(l_1)$ and $\eta(l_5)$ are independent and both are orthogonal to $\eta(l_i, l_5)$, $i \leq 4$, and $\eta(l_k, l_i)$, $i \geq 5$. The reason for this and for many similar such statements in this proof is Lemma 11. Also $\eta(l_i, l_5)$, $i \leq 4$, are orthogonal to each other and nonzero. $\eta(l_k, l_i)$, $i \geq 5$, are orthogonal to each other and nonzero. Since the rank is 5, the sets $\{\eta(l_i, l_5), i \leq 4\}$ and $\{\eta(l_k, l_i), i \geq 5\}$ span the same space, $k = 1, 2, 3, 4$.

In order to render the remainder of the proof easier to follow we write out the relations to be proved in the following tableau:

$$\begin{aligned}\eta(l_1, l_5) &= \eta(l_2, l_6) &&= \eta(l_3, l_7) &&= \eta(l_4, l_8)\ ,\\ \eta(l_1, l_6) &= -\eta(l_2, l_5) &&= \eta(l_3, l_8) &&= -\eta(l_4, l_7)\ ,\\ \eta(l_1, l_7) &= -\eta(l_2, l_8) &&= -\eta(l_3, l_5) &&= \eta(l_4, l_6)\ ,\\ \eta(l_1, l_8) &= \eta(l_2, l_7) &&= -\eta(l_3, l_6) &&= -\eta(l_4, l_5)\ .\end{aligned}$$

We will not keep track of the signs but come back to them at the end.

Now $\eta(l_1, l_5)$, $\eta(l_1, l_6)$, $\eta(l_1, l_7)$, $\eta(l_1, l_8)$ are orthogonal and $\eta(l_2, l_5)$, $\eta(l_2, l_6)$, $\eta(l_2, l_7)$, $\eta(l_2, l_8)$ are orthogonal and span the same space as the first set. Also $\eta(l_1, l_5)$ is orthogonal to $\eta(l_2, l_5)$. We leave $l_1 l_2 l_3 l_4 l_5$ alone and rotate $l_6 l_7 l_8$ among themselves in order to make $\eta(l_2, l_6)$ coincide with $\eta(l_1, l_5)$. We are still free to rotate l_7, l_8 among themselves. From Lemma 5 we obtain

$$\eta(l_1, l_5)\cdot\eta(l_2, l_6) + \eta(l_1, l_2)\cdot\eta(l_5, l_6) + \eta(l_1, l_6)\cdot\eta(l_2, l_5) = 0\ .$$

Since $\eta(l_1, l_2) = 0$ and $\eta(l_1, l_5)\cdot\eta(l_2, l_6) = \eta(l_1, l_5)^2 = \frac{1}{4}$, we have $\eta(l_1, l_6)\cdot\eta(l_2, l_5) = -\frac{1}{4}$. So $\eta(l_2, l_5) = \pm\eta(l_1, l_6)$. Thus $\eta(l_2, l_7), \eta(l_2, l_8)$ being orthogonal to $\eta(l_2, l_5)$ and $\eta(l_2, l_6)$ are also orthogonal to $\eta(l_1, l_5), \eta(l_1, l_6)$ and hence in the same plane as $\eta(l_1, l_7), \eta(l_1, l_8)$. Since $\eta(l_1, l_8)$ and $\eta(l_2, l_7)$ are both orthogonal to $\eta(l_1, l_7)$, $\eta(l_1, l_8) = \pm\eta(l_2, l_7)$. This leaves $\eta(l_1, l_7) = \pm\eta(l_2, l_8)$. We have done the first two columns of the tableau except for signs. We are still free to rotate $l_7 l_8$ in their plane.

Now $\eta(l_3, l_7)$ is orthogonal to $\eta(l_2, l_7)$, hence to $\eta(l_1, l_8)$, and also to $\eta(l_1, l_7)$. Hence it lies in the plane of $\eta(l_1, l_5)$ and $\eta(l_1, l_6)$. Also $\eta(l_3, l_8)$ is orthogonal to $\eta(l_2, l_8)$, hence to $\eta(l_1, l_7)$, and also to $\eta(l_1, l_8)$. Hence it lies in the plane of $\eta(l_1, l_5)$ and $\eta(l_1, l_6)$.

We now perform a rotation of $l_7 l_8$ which leaves $\eta(l_1, l_5)$ and $\eta(l_1, l_6)$ alone and rotates $\eta(l_3, l_7), \eta(l_3, l_8)$ so that $\eta(l_3, l_7)$ coincides with $\eta(l_1, l_5)$. We then have $\eta(l_3, l_8)$ and $\eta(l_1, l_6)$ in the same direction.

Now $\eta(l_4, l_8)$ is orthogonal to $\eta(l_3, l_8)$ and so to $\eta(l_1, l_6)$. It is orthogonal to $\eta(l_2, l_8)$ and so to $\eta(l_1, l_7)$. Since it is also orthogonal to $\eta(l_1, l_8)$, it must lie along $\eta(l_1, l_5)$. Also $\eta(l_4, l_7)$ is orthogonal to $\eta(l_3, l_7)$ and so to $\eta(l_1, l_5)$. It is orthogonal to $\eta(l_2, l_7)$, so to $\eta(l_1, l_8)$, and of course to $\eta(l_1, l_7)$. Hence $\eta(l_4, l_7)$ must lie along $\eta(l_1, l_6)$. We have now completed the first two rows of the tableau as well.

From Lemma 5 we know

$$\eta(l_1, l_8)\cdot\eta(l_3, l_5) + \eta(l_1, l_3)\cdot\eta(l_5, l_8) + \eta(l_1, l_5)\cdot\eta(l_3, l_8) = 0\ .$$

Hence using what we have proved so far we have $\eta(l_1, l_8)\cdot\eta(l_3, l_5) = 0$. So $\eta(l_3, l_5)$ is orthogonal to $\eta(l_1, l_8)$, to $\eta(l_2, l_5)$ and so to $\eta(l_1, l_6)$, and to $\eta(l_1, l_5)$. Hence $\eta(l_3, l_5)$ must lie along $\eta(l_1, l_7)$. The remainder now fills in easily to obtain the entire set of relations up to signs.

To compute the signs we use

$$\eta(l_i, l_j)\cdot\eta(l_k, l_m) + \eta(l_i, l_k)\cdot\eta(l_j, l_m) + \eta(l_i, l_m)\cdot\eta(l_j, l_k) = 0$$

from Lemma 5. The choices of i, j, k, m which are not already zero are 1256, 1278, 3478, 3456, 1357, 1368, 2468, 2457, 1458, 2358, 1467, 2367. In addition we may reflect sending $l_1 \to -l_1$ or $l_i \to -l_i$, $i = 5, 6, 7, 8$, if we wish. In this way we obtain a basis which satisfies the relations of the lemma exactly.

Lemma 16. *Suppose $a = 4$. Let L_1, L_2, L_3 be completely orthogonal subspaces of dimension 4, closed with respect to α. Any basis of $L_1 \oplus L_2$ which respects α in which the relations of Lemma 15 are satisfied may be extended to a basis of $L_1 \oplus L_2 \oplus L_3$ so that the relations are satisfied on $L_1 \oplus L_3$. Furthermore in any basis of $L_1 \oplus L_2 \oplus L_3$ which respects α, if the relations of Lemma 15 are satisfied on $L_1 \oplus L_2$ and $L_1 \oplus L_3$ they are also satisfied on $L_2 \oplus L_3$.*

Proof. Let l_9 be a unit vector in L_3, $l_1 l_2 l_3 l_4$ a basis for L_1, and $l_5 l_6 l_7 l_8$ a basis for L_2 chosen so that the relations of Lemma 15 are satisfied on $L_1 \oplus L_2$. Since $l_5, l_9 \in L_1^\perp$, $l_5 \cos\theta + l_9 \sin\theta \in L_1^\perp$. Let $L(\theta) = \alpha(l_5 \cos\theta + l_9 \sin\theta)$. By Lemma 10, $L(\theta)$ and L_1 are completely orthogonal. By Lemma 12, $L(\theta) \oplus L_1$ is closed with respect to α. By applying Lemma 15 to $L(\theta) \oplus L_1$, we see that the basis provided by the lemma is continuous in θ. Hence no reflection in L_1 can occur. Thus we may find a basis $l_9 l_{10} l_{11} l_{12}$ so that on $L_1 \oplus L_3$ the relations of Lemma 15 are satisfied in the basis $l_1 l_2 l_3 l_4 l_9 l_{10} l_{11} l_{12}$.

Now we show that in the basis $l_5, \cdots, l_{12}$ the relations of Lemma 15 are satisfied on $L_1 \oplus L_3$. First, as in the proof of Lemma 15, by computing the rank at $l_5 + l_9$, we see that

$$\begin{aligned} &\eta(l_5, l_9) \wedge \eta(l_5, l_{10}) \wedge \eta(l_5, l_{11}) \wedge \eta(l_5, l_{12}) \\ &\qquad = \lambda \eta(l_5, l_9) \wedge \eta(l_6, l_9) \wedge \eta(l_7, l_9) \wedge \eta(l_8, l_9) \end{aligned}$$

for $\lambda \neq 0$. Then on $L_1 \oplus L_2 \oplus L_3$ the rank of the Jacobian of η falls by 3. Thus among η_{x_i}, $i = 1, \cdots, 12$, any ten are dependent. We evaluate the Jacobian at $l_1 + l_5 + tl_9$, $t \neq 0$. The vectors $\eta_{x_1}, \eta_{x_5}, \eta_{x_9}$ are independent from each other and from all the other η_{x_i}. This is because they have, respectively, terms $\eta(l_1)$, $\eta(l_5), \eta(l_9)$ and η_{x_i}, $i \neq 1, 5, 9$, are sums of terms of the form $\eta(l_i, l_j)$, $i \neq j$. Since $\eta(l_1), \eta(l_5), \eta(l_9)$ are orthogonal to all these vectors, they must be independent of them. Furthermore, $\eta(l_1), \eta(l_5), \eta(l_9)$ are all unit vectors and the inner product of any two is $\frac{1}{2}$. Since no such triple of vectors can be linearly dependent, among $\eta_{x_2}, \eta_{x_3}, \eta_{x_4}, \eta_{x_6}, \eta_{x_7}, \eta_{x_8}, \eta_{x_{10}}, \eta_{x_{11}}, \eta_{x_{12}}$ any seven are dependent.

Let $\Lambda(t) = \eta_{x_2} \wedge \eta_{x_3} \wedge \eta_{x_4} \wedge \eta_{x_6} \wedge \eta_{x_7} \wedge \eta_{x_8}$ evaluated at $l_1 + l_5 + tl_9$ and let

$$\Lambda_{10} = \Lambda(t) \wedge \eta_{x_{10}}, \quad \Lambda_{11} = \Lambda(t) \wedge \eta_{x_{11}}, \quad \Lambda_{12} = \Lambda(t) \wedge \eta_{x_{12}}.$$

Then $\Lambda_{10}, \Lambda_{11}, \Lambda_{12}$ must be identically zero. Ostensibly they are of sixth degree in t; however by computing the rank at $l_5 + l_9$ we see that the highest degree term is 0 because $\eta(l_5, l_{10}), \eta(l_5, l_{11}), \eta(l_5, l_{12})$ lie in the span of $\eta(l_6, l_9), \eta(l_7, l_9)$, $\eta(l_8, l_9)$ as was stated above. Then compute the 5th degree terms of $\Lambda_{10}, \Lambda_{11}, \Lambda_{12}$

using the fact that the relations of Lemma 15 are satisfied on $L_1 \oplus L_2$ and $L_1 \oplus L_3$. By equating these terms to zero we find that

$$\eta(l_5, l_{10}) = -\eta(l_6, l_9) , \quad \eta(l_5, l_{11}) = -\eta(l_7, l_9) , \quad \eta(l_5, l_{12}) = -\eta(l_8, l_9) .$$

This does not give us quite enough information, so we now evaluate the Jacobian at $l_1 + l_5 + tl_{10}$ and proceed as before. This time we may disregard $\eta_{x_1}, \eta_{x_5}, \eta_{x_{10}}$ and of the remaining, any seven must be dependent. We choose $\eta_{x_2} \wedge \eta_{x_3} \wedge \eta_{x_4} \wedge \eta_{x_6} \wedge \eta_{x_7} \wedge \eta_{x_8} \wedge \eta_{x_{11}}$. We compute the fifth degree term in t and equate it to 0 obtaining

$$\eta(l_5, l_{11}) = \eta(l_8, l_{10}) .$$

We next use

$$\eta(l_i, l_j)\cdot\eta(l_k, l_m) + \eta(l_i, l_k)\cdot\eta(l_j, l_m) + \eta(l_i, l_m)\cdot\eta(l_j, l_k) = 0$$

from Lemma 5 and the fact that if $\eta(l_i, l_j)\cdot\eta(l_k, l_m) = \pm\frac{1}{4}$ then $\eta(l_i, l_j) = \pm\eta(l_k, l_m)$, respectively because $|\eta(l_i, l_j)| = |\eta(l_k, l_m)| = \frac{1}{2}$. This enables us to complete the proof that all relations of Lemma 15 are satisfied on $L_2 \oplus L_3$.

As an example of the computations we show that $\eta(l_5, l_{10}) = -\eta(l_6, l_9)$. Taking into accocnt that the relations of Lemma 15 are satisfied on $L_1 \oplus L_2$ and $L_1 \oplus L_3$ we obtain η_{x_i} evaluated at $l_1 + l_5 + tl_9$:

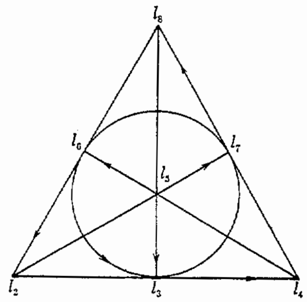
$$\begin{aligned}
&\eta_{x_2} = -\eta(l_1, l_6) - t\eta(l_1, l_{10}) , &&\eta_{x_3} = -\eta(l_1, l_7) - t\eta(l_1, l_{11}) , \\
&\eta_{x_4} = -\eta(l_1, l_8) - t\eta(l_1, l_{12}) , &&\eta_{x_6} = \eta(l_1, l_6) + t\eta(l_6, l_9) , \\
&\eta_{x_7} = \eta(l_1, l_7) + t\eta(l_7, l_9) , &&\eta_{x_8} = \eta(l_1, l_8) + t\eta(l_8, l_9) , \\
&\qquad\qquad \eta_{x_{10}} = \eta(l_1, l_{10}) + \eta(l_5, l_{10}) .
\end{aligned}$$

We write $\eta(l_5, l_{10}) = a\eta(l_6, l_9) + b\eta(l_7, l_9) + c\eta(l_8, l_9)$. The t^5 term of the wedge of the above seven vectors after simplification is

$$\begin{aligned}
\eta(l_1, l_{10}) \wedge \eta(l_1, l_{11}) \wedge \eta(l_1, l_{12}) \wedge \eta(l_6, l_9) \wedge \eta(l_7, l_9) \wedge \eta(l_8, l_9) \\
\wedge [\eta(l_1, l_6) + a\eta(l_1, l_6) + b\eta(l_1, l_7) + c\eta(l_1, l_8)] .
\end{aligned}$$

Now $\eta(l_1, l_6)$, $\eta(l_1, l_7)$, $\eta(l_1, l_8)$, $\eta(l_1, l_{10})$, $\eta(l_1, l_{11})$, $\eta(l_1, l_{12})$, $\eta(l_6, l_9)$, $\eta(l_7, l_9)$, $\eta(l_8, l_9)$ are all nonzero and orthogonal to each other. Use Lemma 5 and the relations of Lemma 15 satisfied on $L_1 \oplus L_2$ and $L_1 \oplus L_3$ to show orthogonality. Since this must be zero we see that $1 + a = b = c = 0$ and $\eta(l_5, l_{10}) = -\eta(l_6, l_9)$.

Remark. Cayley multiplication on a basis $l_1, \cdots, l_8$ of E^8 may be defined as follows. Let $l_2, \cdots, l_8$ be the seven points of a projective plane over Z_2 with cyclic ordering of each line given as in the figure:



Define $-l_j l_i = l_i l_j = l_k$ in case $l_i l_j l_k$ has the given cyclic ordering. Define further $l_i^2 = -l_1$, $i \neq 1$, and $l_1 l_i = l_i l_1 = l_i$ for all i. For this definition see Freudenthal [1]. Conjugation is defined by $\bar{l}_1 = l_1$, $\bar{l}_i = -l_i$, $i = 2, \cdots, 8$.

Lemma 17. *Suppose $a = 8$. Let L_1, L_2 be two completely orthogonal subspaces of dimension 8 closed with respect to α. Then there are bases $l_1 \cdots l_8$ of L_1 and $l_9 \cdots l_{16}$ of L_2 so that $\eta(l_i, l_{j+8}) = \pm\eta(l_k, l_{m+8})$ if and only if in the Cayley product given above $l_i \bar{l}_j = \pm l_k \bar{l}_m$. Here both signs are taken as positive or both as negative, and the indicies range from 1 to 8.*

Proof. Let $l_1 \cdots l_8$ be an orthonormal basis of L_1 and $l_9 \cdots l_{16}$ of L_2. By Lemma 12, η falls in rank by 7 on $L_1 \oplus L_2$. Now

$$\eta_{x_i} = 2 \sum_{j=1}^{16} x_j \eta(l_i, l_j) .$$

Fix $k \leq 8$ and $m \geq 9$. At $l_k + l_m$

$$\begin{aligned} \eta_{x_i} &= 2\eta(l_i, l_m) , & i \leq 8, i \neq k , \\ \eta_{x_i} &= 2\eta(l_i, l_k) , & i \geq 9, i \neq m , \\ \eta_{x_k} &= 2\eta(l_k) + 2\eta(l_k, l_m) , & \\ \eta_{x_m} &= 2\eta(l_m) + 2\eta(l_k, l_m) . & \end{aligned}$$

Now $\eta(l_k)$, $\eta(l_m)$ are orthogonal to η_{x_i}, $i \neq k, m$, and to $\eta(l_k, l_m)$. They are unit vectors and independent since $\eta(l_k) \cdot \eta(l_m) = \frac{1}{2}$. Thus η_{x_k}, η_{x_m} are not dependent on η_{x_i}, $i \neq k, m$, and therefore any 8 of η_{x_i}, $i \neq k, m$, must be dependent. But $\eta(l_i, l_m)$ for $i \leq 8$, $i \neq k$, are orthogonal and hence independent. Thus $\eta(l_i, l_k)$ for $i \geq 9$, $i \neq m$ depends on $\{\eta(l_i, l_m), i \leq 8, i \neq k\}$. Similarly $\eta(l_i, l_m)$, $i \leq 8$, $i \neq k$, depends on $\{\eta(l_i, l_k), i \geq 9, i \neq m\}$. Hence the sets

$$\{\eta(l_i, l_m), i \leq 8\} \quad \text{and} \quad \{\eta(l_i, l_k), i \geq 9\} ,$$

for any $m \geq 9$, $k \leq 8$, all span the same space.

We write out the relations to be proved to make it easier to follow the arguments. Because the list is large we abbreviate $\eta(l_i, l_j)$ by i, j and $-\eta(l_i, l_j)$ by

$-i, j$. We also leave out the equal signs because we understand that the vectors in each row are equal. The tableau of relation is:

1, 9	2, 10	3, 11	4, 12	5, 13	6, 14	7, 15	8, 16
1, 10	−2, 9	3, 12	−4, 11	5, 15	−6, 16	−7, 13	8, 14
1, 11	−2, 12	−3, 9	4, 10	−5, 16	−6, 15	7, 14	8, 13
1, 12	2, 11	−3, 10	−4, 9	5, 14	−6, 13	7, 16	−8, 15
1, 13	−2, 15	3, 16	−4, 14	−5, 9	6, 12	7, 10	−8, 11
1, 14	2, 16	3, 15	4, 13	−5, 12	−6, 9	−7, 11	−8, 10
1, 15	2, 13	−3, 14	−4, 16	−5, 10	6, 11	−7, 9	8, 12
1, 16	−2, 14	−3, 13	4, 15	5, 11	6, 10	−7, 12	−8, 9

$\{\eta(l_2, l_i),\ i = 9, \cdots, 16\}$ and $\{\eta(l_1, l_i),\ i = 9, \cdots, 16\}$ are each sets of orthogonal vectors spanning the same space. Furthermore $\eta(l_1, l_9)$ and $\eta(l_2, l_9)$ are orthogonal. As $l_{10}, \cdots, l_{16}$ rotate among themselves, $\eta(l_2, l_{10})$ is carried into any vector orthogonal to $\eta(l_2, l_9)$. In particular we may rotate so that $\eta(l_1, l_9) = \eta(l_2, l_{10})$. Then using Lemma 5 we find $\eta(l_1, l_{10}) = -\eta(l_2, l_9)$. During this proof each use of Lemma 5 refers to the formula:

$$\eta(l_i, l_j)\cdot\eta(l_k, l_m) + \eta(l_i, l_k)\cdot\eta(l_j, l_m) + \eta(l_i, l_m)\cdot\eta(l_j, l_k) = 0 \ ,$$

where we use $i, j, k, m = 1, 2, 9, 10$.

Now $\{\eta(l_1, l_i),\ i = 11, \cdots, 16\}$ and $\{\eta(l_2, l_i),\ i = 11, \cdots, 16\}$ are orthogonal sets spanning the same space. Furthermore $\eta(l_1, l_{11})$ and $\eta(l_2, l_{11})$ are orthogonal. Hence by rotating $l_{12}, \cdots, l_{16}$ among themselves we may achieve $\eta(l_1, l_{11}) = -\eta(l_2, l_{12})$. By Lemma 5, $\eta(l_1, l_{12}) = \eta(l_2, l_{11})$. Again $\{\eta(l_1, l_i),\ i = 13, \cdots, 16\}$ and $\{\eta(l_2, l_i)\ i = 13, \cdots, 16\}$ are orthogonal vectors spanning the same space. Since $\eta(l_1, l_{13})$ and $\eta(l_2, l_{13})$ are orthogonal, rotating l_{14}, l_{15}, l_{16} among themselves we achieve $\eta(l_1, l_{13}) = -\eta(l_2, l_{15})$. By Lemma 5, $\eta(l_1, l_{15}) = \eta(l_2, l_{13})$. This leaves $\{\eta(l_1, l_{14}),\ \eta(l_1, l_{16})\}$ and $\{\eta(l_2, l_{14}),\ \eta(l_2, l_{16})\}$ spanning the same plane. But $\eta(l_1, l_{14})$ and $\eta(l_2, l_{14})$ are orthogonal. Hence by changing l_{16} to $-l_{16}$ if necessary we may achieve $\eta(l_1, l_{14}) = \eta(l_2, l_{16})$ and by Lemma 5, $\eta(l_1, l_{16}) = -\eta(l_2, l_{14})$. Thus the first two columns of the tableau are equal.

Since $\eta(l_1, l_{12}) = \eta(l_2, l_{11})$ we see that $\{\eta(l_i, l_{11}),\ i = 3, \cdots, 8\}$ and $\{\eta(l_1, l_i),\ i = 9, 10, 13, 14, 15, 16\}$ span the same space. We rotate $l_3, \cdots, l_8$ among themselves to make $\eta(l_1, l_9) = \eta(l_3, l_{11})$. Apply Lemma 5 to $\eta(l_1, l_9) = \eta(l_2, l_{10}) = \eta(l_3, l_{11})$ to see that $\eta(l_2, l_{11}) = -\eta(l_3, l_{10})$ and $\eta(l_1, l_{11}) = -\eta(l_3, l_9)$. Hence $-\eta(l_2, l_{12}) = -\eta(l_3, l_9)$ and Lemma 5 applied to this gives $-\eta(l_2, l_9) = \eta(l_3, l_{12})$.

The conditions $\eta(l_1, l_{14}) = \eta(l_2, l_{16})$ and $\eta(l_1, l_{16}) = -\eta(l_2, l_{14})$ are preserved by a rotation of l_{14}, l_{16} in their plane. Hence we may yet rotate l_{14}, l_{16} and not change any of the relations so far established. But $\eta(l_3, l_{14})$ and $\eta(l_3, l_{16})$ lie in the plane spanned by $\eta(l_1, l_{13})$ and $\eta(l_1, l_{15})$. Thus performing a rotation of l_{14}, l_{16} we may achieve $\eta(l_1, l_{13}) = \eta(l_3, l_{16})$. By Lemma 5, $\eta(l_1, l_{16}) = -\eta(l_3, l_{13})$. So the first three columns of the tableau are equal.

Now $\{\eta(l_i, l_{12}), i = 4, \cdots, 8\}$ and $\{\eta(l_1, l_i), i = 9, 13, 14, 15, 16\}$ both span the same space. Hence by rotating $l_4, l_5, \cdots, l_8$ we may achieve $\eta(l_1, l_9) = \eta(l_4, l_{12})$. Using the fact that $\eta(l_1, l_9) = \eta(l_2, l_{10}) = \eta(l_3, l_{11}) = \eta(l_4, l_{12})$ and applying Lemma 5 we see that $\eta(l_1, l_{12}) = -\eta(l_4, l_9)$, $-\eta(l_2, l_{12}) = \eta(l_4, l_{10})$ and $\eta(l_3, l_{12}) = -\eta(l_4, l_{11})$. But now $\{\eta(l_4, l_i), i = 13, \cdots, 16\}$ and $\{\eta(l_1, l_i), i = 13, \cdots, 16\}$ span the same space. Because $\eta(l_1, l_{15}) = -\eta(l_3, l_{14})$ and $\eta(l_1, l_{16}) = -\eta(l_2, l_{14})$ we see that $\eta(l_4, l_{14})$ is orthogonal to $\eta(l_1, l_i)$, $i = 14, 15, 16$. Hence $\eta(l_1, l_{13}) = -\lambda\eta(l_4, l_{14})$, $\lambda = \pm 1$. By Lemma 5 because $\eta(l_1, l_{13}) = -\eta(l_2, l_{15}) = \eta(l_3, l_{16}) = -\lambda\eta(l_4, l_{14})$ we see that $\eta(l_1, l_{14}) = \lambda\eta(l_4, l_{13})$, $-\eta(l_2, l_{14}) = \lambda\eta(l_4, l_{15})$ and $-\eta(l_3, l_{14}) = -\lambda\eta(l_4, l_{16})$. Hence except for the determination of λ, the first four columns of the tableau are equal.

Now $\{\eta(l_i, l_{13}), i = 5, 6, 7, 8\}$ lies in the span of $\{\eta(l_1, l_i), i = 9, 10, 11, 12\}$. By rotating among $l_5 l_6 l_7 l_8$ we may assume that $\eta(l_1, l_9) = \eta(l_5, l_{13})$. We apply Lemma 5 successively to a list of relations each of which is true by an application of Lemma 5 to an earlier member of the list and use of the fact that the first four columns in the tableau are equal, except for λ. The list is $\eta(l_1, l_9) = \eta(l_5, l_{13})$; $\eta(l_3, l_{11}) = \eta(l_5, l_{13})$; $-\eta(l_2, l_{14}) = \eta(l_5, l_{11})$; $-\eta(l_4, l_9) = \eta(l_5, l_{14})$; $\eta(l_1, l_{12}) = \eta(l_5, l_{14})$; $-\eta(l_3, l_{10}) = \eta(l_5, l_{14})$; $\eta(l_1, l_{16}) = \eta(l_5, l_{11})$; and $\eta(l_1, l_{15}) = -\eta(l_5, l_{10})$. The result is that $\lambda = +1$ and the first five columns of the tableau are equal.

Now $\{\eta(l_i, l_{14}), i = 6, 7, 8\}$ and $\{\eta(l_1, l_i), i = 9, 10, 11\}$ span the same space. Hence by rotating $l_6 l_7 l_8$ we may make $\eta(l_1, l_9) = \eta(l_6, l_{14})$. We apply Lemma 5 to the relations of the first row as far as we know them and then to $\eta(l_1, l_{16}) = \eta(l_4, l_{15}) = \eta(l_6, l_{10})$ to conclude that the first six columns of the tableau are equal.

Again $\{\eta(l_i, l_{15}), i = 7, 8\}$ and $\{\eta(l_1, l_9), \eta(l_1, l_{12})\}$ span the same plane. Thus rotating $l_7 l_8$ we may achieve $\eta(l_1, l_9) = \eta(l_7, l_{15})$. Applying Lemma 5 to the relations of the first row as far as we know them and then to $\eta(l_1, l_{16}) = -\eta(l_7, l_{12})$ we conclude that the first seven columns of the tableau are equal.

By sending l_8 to $-l_8$ if necessary we see that $\eta(l_1, l_9) = \eta(l_8, l_{16})$. Applying Lemma 5 to the relations of the first row finishes the proof.

Proof of the theorem. We may assume by Proposition 4 that all the geodesics of M are circles of radius 1. The unit tangent sphere TS_p^{n-1} is fibred by great spheres of dimensions $a - 1$. Namely the point $l \in TS_p^{n-1}$ lies on the great sphere $\alpha(l) \cap TS_p^{n-1}$. By Proposition 13 they all have the same dimension $a - 1$. But it is a theorem of topology that an $(n - 1)$-sphere can be fibred by spheres of dimension $a - 1$ only if $a = 1, 2, 4, 8$, or n. For $a = 1, 2$ or 4, n may be any multiple of 1, 2, or 4 respectively. $a = n$ may hold for any n and the only other case is $a = 8$ and $n = 16$.

If $a = n$ then M is a unit n-sphere because all the geodesics through a point have the same center. (See the remark after Proposition 7.) For the other cases where $\alpha = 1, 2, 4$ or 8 we use Lemma 11 and Lemmas 14–17 to find a basis l_i of T_p such that $\eta(l_i, l_j) \cdot \eta(l_k, l_m)$ are known for all i, j, k, m.

In the cases where $a = 1, 2, 4$ or 8 let V be the given embeddings of RP^n, CP^n, LP^n, OP^2 respectively. Perform a dilatation of the Euclidean space so that the geodesics of V have radius 1, and assume V and M lie in the same Euclidean space.

Now V is a manifold with planar geodesics. Hence by our previous calculations we may find a basis l_{iV} of T_pV such that the quantities $\eta_V(l_{iV}, l_{jV}) \cdot \eta_V(l_{kV}, l_{mV})$ have the calculated values.

Perform a translation to make M and V coincide at one point. Then perform a rotation about that point to make the tangent planes of M and V coincide at that one point. Let l_i be the basis in the common tangent plane in which $\eta(l_i, l_j) \cdot \eta(l_k, l_m)$ were computed, and l_{iV} the corresponding basis for V. Rotate and reflect about the common point until l_i coincides with l_{iV}.

Now if two sets of vectors have identical inner products (for corresponding pairs), we may perform a rotation and reflection about the origin to make them agree. Using this fact we may perform a rotation and reflection in the normal space, leaving the common tangent plane pointwise fixed to make $\eta_V(l_i, l_j) = \eta(l_i, l_j)$. This implies that $\eta = \eta_V$ at that point. Hence the geodesics of each manifold through that point coincide so that the manifolds coincide locally. By analytic continuation M is either an open subset of an n-plane or congruent to a dilatation of an open subset of a manifold in the list.

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